Eulerian Video Magnification for Revealing Subtle Changes in the World
Supplemental

1 Eulerian and Lagrangian Motion Magnification Error - Detailed Derivation

Here we give the derivation in Appendix A in the paper in more detail.

In this section we derive estimates of the error in the Eulerian and Lagrangian motion magnification results with respect to spatial and temporal noise. The derivation is done again for the 1D case for simplicity, and can be generalized to 2D. We use the same setup as in Sect. 3.1 in the paper, where the true motion-magnified sequence is

$$\begin{aligned} \hat{I}(x,t) &= f(x + (1+\alpha)\delta(t)) \\ &= I(x + (1+\alpha)\delta(t), 0) \end{aligned} \tag{1}$$

Both methods only approximate the true motion-amplified sequence, $\hat{I}(x,t)$ (Eq. 1). Let us first analyze the error in those approximations on the clean signal, $I(x,t)$.

1.1 Without Noise

Lagrangian. In the Lagrangian approach, the motion-amplified sequence, $\tilde{I}_L(x,t)$, is achieved by directly amplifying the estimated motion, $\tilde{\delta}(t)$, with respect to the reference frame $I(x,0)$

$$\tilde{I}_L(x,t) = I(x + (1+\alpha)\tilde{\delta}(t), 0) \tag{2}$$

In its simplest form, we can estimate $\delta(t)$ using point-wise brightness constancy (See the paper for discussion on spatial regularization)

$$\tilde{\delta}(t) = \frac{I_t(x,t)}{I_x(x,t)} \tag{3}$$

where $I_x(x,t) = \partial I(x,t)/\partial x$ and $I_t(x,t) = I(x,t) - I(x,0)$. From now on, we will omit the space (x) and time (t) indices when possible for brevity.

The error in in the Lagrangian solution is directly determined by the error in the estimated motion, which we take to be second-order term in the brightness constancy equation

$$\begin{aligned} I(x,t) &= I(x + \delta(t), 0) \\ &\approx I(x,0) + \delta(t) I_x + \frac{1}{2}\delta^2(t) I_{xx} \\ \frac{I_t}{I_x} &\approx \delta(t) + \frac{1}{2}\delta^2(t) I_{xx} \end{aligned} \tag{4}$$

So that the estimated motion $\tilde{\delta}(t)$ is related to the true motion, $\delta(t)$, as

$$\tilde{\delta}(t) \approx \delta(t) + \frac{1}{2}\delta^2(t) I_{xx} \tag{5}$$

Plugging (5) in (2),

$$\begin{aligned} \tilde{I}_L(x,t) &\approx I\left(x + (1+\alpha)\left(\delta(t) + \frac{1}{2}\delta^2(t) I_{xx}\right), 0\right) \\ &\approx I\left(x + (1+\alpha)\delta(t) + \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx}, 0\right) \end{aligned} \tag{6}$$

Using first-order Taylor expansion of I about $x + (1+\alpha)\delta(t)$,

$$\tilde{I}_L(x,t) \approx I(x + (1+\alpha)\delta(t), 0) + \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x \tag{7}$$

Subtracting (1) from (7), the error in the Lagrangian motion-magnified sequence, ε_L, is

$$\varepsilon_L \approx \left|\frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x\right| \tag{8}$$

Eulerian. In our Eulerian approach, the magnified sequence, $\hat{I}_E(x,t)$, is computed as

$$\begin{aligned}\tilde{I}_E(x,t) &= I(x,t) + \alpha I_t(x,t) \\ &= I(x,0) + (1+\alpha) I_t(x,t)\end{aligned} \tag{9}$$

similar to Eq. 4 in the paper, using a two-tap temporal filter to compute I_t.

Using Taylor expansion of the true motion-magnified sequence, $\hat{I}$ (Eq. 1), about x, we have

$$\hat{I}(x,t) \approx I(x,0) + (1+\alpha)\delta(t) I_x + \frac{1}{2}(1+\alpha)^2 \delta^2(t) I_{xx} \tag{10}$$

Plugging (4) into (10)

$$\begin{aligned}\hat{I}(x,t) &\approx I(x,0) + (1+\alpha)(I_t - \frac{1}{2}\delta^2(t) I_{xx} I_x) + \frac{1}{2}(1+\alpha)^2 \delta^2(t) I_{xx} \\ &\approx I(x,0) + (1+\alpha) I_t - \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x + \frac{1}{2}(1+\alpha)^2 \delta^2(t) I_{xx}\end{aligned} \tag{11}$$

Subtracting (9) from (11) gives the error in the the Eulerian solution

$$\varepsilon_E \left| \approx \frac{1}{2}(1+\alpha)^2 \delta^2(t) I_{xx} - \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x \right| \tag{12}$$

1.2 With Noise

Let $I'(x,t)$ be the noisy signal, such that

$$I'(x,t) = I(x,t) + n(x,t) \tag{13}$$

for additive noise $n(x,t)$.

Lagrangian. The estimated motion becomes

$$\tilde{\delta}(t) = \frac{I'_t}{I'_x} = \frac{I_t + n_t}{I_x + n_x} \tag{14}$$

where $n_x = \partial n / \partial x$ and $n_t = n(x,t) - n(x,0)$.

Using Taylor Expansion on (n_t, n_x) about $(0,0)$ (zero noise), and using (4), we have

$$\begin{aligned}\tilde{\delta}(t) &\approx \frac{I_t}{I_x} + n_t \frac{1}{I_x + n_x} + n_x \frac{I_t + n_t}{(I_x + n_x)^2} \\ &\approx \delta(t) + \frac{n_t}{I_x} - n_x \frac{I_t}{I_x^2} + \frac{1}{2}\delta^2(t) I_{xx}\end{aligned} \tag{15}$$

where we ignored the terms involving products of the noise components.

Plugging into Eq. (2), and using Taylor expansion of I about $x + (1+\alpha)\delta(t)$, we get

$$\tilde{I}'_L(x,t) \approx I(x + (1+\alpha)\delta(t), 0) + (1+\alpha) I_x (\frac{n_t}{I_x} - n_x \frac{I_t}{I_x^2} + \frac{1}{2} I_{xx} \delta^2(t))) + n \tag{16}$$

Arranging terms, and Substituting (4) in (16),

$$\begin{aligned}\tilde{I}'_L(x,t) &\approx I(x + (1+\alpha)\delta(t), 0) + (1+\alpha)\left(n_t - n_x\left(\delta(t) + \frac{1}{2}\delta^2(t) I_{xx}\right) + \frac{1}{2}\delta^2(t) I_{xx} I_x\right) + n \\ &= I(x + (1+\alpha)\delta(t), 0) + (1+\alpha) n_t - (1+\alpha) n_x \delta(t) - \frac{1}{2}(1+\alpha) n_x \delta^2(t) I_{xx} + \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x + n\end{aligned} \tag{17}$$

Using (5) again and subtracting (1), the Lagrangian error as function of noise, $\varepsilon_L(n)$, is

$$\varepsilon_L(n) \approx \left| (1+\alpha) n_t - (1+\alpha) n_x \delta(t) - \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} n_x + \frac{1}{2}(1+\alpha)\delta^2(t) I_{xx} I_x + n \right| \tag{18}$$

Eulerian. The noisy motion-magnified sequence becomes

$$\begin{aligned}\tilde{I}'_E(x,t) &= I'(x,0) + (1+\alpha)I'_t \\ &= I(x,0) + (1+\alpha)(I_t + n_t) + n \\ &= I_E(x,t) + (1+\alpha)n_t + n\end{aligned} \tag{19}$$

Using (12) and subtracting (1), the Eulerian error as function of noise, $\varepsilon_E(n)$, is

$$\varepsilon_E(n) \approx \left|(1+\alpha)n_t + \frac{1}{2}(1+\alpha)^2\delta^2(t)I_{xx} - \frac{1}{2}(1+\alpha)\delta^2(t)I_{xx}I_x + n\right| \tag{20}$$

Notice that setting zero noise in (18) and (20), we get the corresponding errors derived for the non-noisy signal in (8) and (12).